

Steinitz Class of Mordell Groups of Elliptic Curves With Complex Multiplication

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Abstract. Let E be an elliptic curve having Complex Multiplication by the full ring \mathcal{O}_K of integers of $K = \mathbf{Q}(\sqrt{-D})$, let $H = K(j(E))$ be the Hilbert class field of K . Then the Mordell-Weil group $E(H)$ is an \mathcal{O}_K -module, and its Steinitz class $\text{St}(E)$ is studied. When D is a prime number, it is proved that $\text{St}(E) = 1$ if $D \equiv 3 \pmod{4}$; and $\text{St}(E) = [\mathcal{P}]^t$ if $p \equiv 1 \pmod{4}$, where $[\mathcal{P}]$ is the ideal class of K represented by prime factor \mathcal{P} of 2 in K , t is a fixed integer. General structures are also discussed for $\text{St}(E)$ and for modules over Dedekind domain. These results develop the results by D. Dummit and W. Miller for $D = 10$ and some elliptic curves to more general D and elliptic curves.

Keywords: elliptic curve, Mordell group, complex multiplication, Steinitz class, module

1 introduction

Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic number field, \mathcal{O}_K the ring of all integers of K . Let E be an elliptic curve having Complex Multiplication by the ring \mathcal{O}_K . E is defined over the field $F = \mathbf{Q}(j(E))$, $j(E)$ is the j -invariant of E . So $H = K(j(E))$ is the Hilbert class of K ^[3], and the Mordell-Weil group $E(H)$ of H -rational points of E is then naturally a module over the Dedekind domain \mathcal{O}_K (via complex multiplication). By the structure theorem for finitely generated modules over Dedekind domain we have that

$$E(H) \cong E(H)_{\text{tor}} \oplus \mathcal{O}_K \oplus \cdots \oplus \mathcal{O}_K \oplus \mathcal{A} = E(H)_{\text{tor}} \oplus \mathcal{O}_K^{s-1} \oplus \mathcal{A},$$

here \mathcal{A} is an ideal of \mathcal{O}_K which is uniquely determined up to a multiplication by a number from K . Thus $E(H)$ uniquely defines an ideal class $[\mathcal{A}]$ of K represented by \mathcal{A} , which is said to be the Steinitz class of E and denoted by $St(E)$. (Similarly, any module M over a Dedekind domain R defines an ideal class of R , which is said to be the Steinitz class of M and denoted by $St(M)$.) So the structure of the Mordell group $E(H)$, as a module over the Dedekind domain \mathcal{O}_K , is uniquely determined by its rank s , Steinitz class $St(E)$, and torsion part. Therefore, it is important to determine the Steinitz class $St(E)$. D. Dummit and W. Miller ^[5] in 1996 determined the Steinitz class for some specific elliptic curves when $D = 10$ and also found some properties of them.

Let $l = rank_{\mathbf{Z}}(E(F))$ be the \mathbf{Z} -rank of $(E(F))$, $G = Gal(H/F)$ be the Galois group of H/F (which will be shown to be a quadratic extension). Let $[\mathcal{A}]$ denote the ideal class represented by the ideal \mathcal{A} of \mathcal{O}_K . Since $St(E)$ is concerned only with the free part of $E(H)$, we put $E(\cdot)_f = E(\cdot)/E(\cdot)_{tor}$, i.e., the quotient group of the Mordell group $E(\cdot)$ modulo its torsion part. Note that $E(\cdot)_f$ is isomorphic to the free part of $E(\cdot)$. We will also use this notation to subgroups of Mordell groups.

We will analysis the interior structure of $E(H)$, give a general theorem for the structure of modules over Dedekind domain, and then determine Steinitz classes $St(E)$ for some types of elliptic curves. In particular, when $D = p$ is a prime number and $p \equiv 3 \pmod{4}$, we will prove that $St(E)$ is the principal class of K . And when the prime number $D = p \equiv 1 \pmod{4}$, we will show that

$$St(E) = [\mathcal{P}]^t, \quad t = l + \log |H^1(G, E(H)_f)|$$

where $|H^1(G, E(H)_f)|$ is the order of the first cohomology group $H^1(G, E(H)_f)$, and \mathcal{P} is any prime factor of 2 in K .

The Weierstrass equation of E could be assumed as ^[3]

$$E : y^2 = f(x) = x^3 + a_2x^2 + a_4x + a_6$$

with $a_2, a_4, a_6 \in F$.

2 Structure of $E(H)$

Lemma 1. The degree of the extension H/F is $[H : F] = 2$.

Proof. Obviously $[H : F] \leq 2$. Now if $[H : F] = 1$, then $K \subset F$. Consider $F = \mathbf{Q}(j(E))$, where E could be any elliptic curve having complex multiplication by \mathcal{O}_K . Since the complex multiplication domain \mathcal{O}_K is a \mathbf{Z} -module of rank 2, so the j -invariant $j(E)$ is a real number^[6]. Thus $F = \mathbf{Q}(j(E))$ has a real embedding into the complex field. Note that K is totally imaginary, $K \subset F$ is impossible, so $H \not\subset F$, $[H : F] = [K(j(E)) : \mathbf{Q}(j(E))] = 2$. This proves the lemma.

For any $\alpha \in \mathcal{O}_K$, let $[\alpha]$ denote the endomorphism of E corresponding to α . Comparing to E , we consider the following elliptic curve

$$E_D : \quad -Dy^2 = f(x).$$

Note that E_D and E are isomorphic via the map

$$i : \quad E_D(\mathbf{C}) \rightarrow E(\mathbf{C}), \quad (x, y) \rightarrow (x, \sqrt{-D}y).$$

Thus E_D also has complex multiplication by \mathcal{O}_K , and is defined over F . Via the isomorphism i of E and E_D , we have obviously that

$$E_D(F) \cong I = \{(x, \sqrt{-D}y) | (x, \sqrt{-D}y) \in E(H), x, y \in F\} \subset E(H).$$

The subgroup I of $E(H)$ defined here is very important in the following analysis.

Lemma 2. The map $i \circ [\sqrt{-D}]$ is an F -isogeny of E to E_D . Thus

$$\text{rank}_{\mathbf{Z}}(E_D(F)) = \text{rank}_{\mathbf{Z}}(E(F)) = l$$

Proof. By [5] we have

$$[\sqrt{-D}](x, y) = (a(x), y\sqrt{-D}b(x)),$$

with $a(x), b(x) \in F(x)$. So $i \circ [\sqrt{-D}]$ is an F -isogeny of E to E_D .

Lemma 3. $(I_f : [\sqrt{-D}]E(F)_f)(E(F)_f : [\sqrt{-D}]I_f) = D^l$

Proof.

$$\begin{aligned}
D^l &= (E(F)_f : [D]E(F)_f) \\
&= (E(F)_f : [\sqrt{-D}]I_f)([\sqrt{-D}]I_f : [D]E(F)_f) \\
&= (E(F)_f : [\sqrt{-D}]I_f)(I_f : [\sqrt{-D}]E(F)_f).
\end{aligned}$$

Lemma 4. $2E(H)_f \subset E(F)_f \oplus I_f \subset E(H)_f$,

$$\text{rank}_{\mathbf{Z}}(E(H)) = \text{rank}_{\mathbf{Z}}(E(F)) + \text{rank}_{\mathbf{Z}}(E_D(F)) = 2 \text{rank}_{\mathbf{Z}}(E(F)) = 2l.$$

Proof. If $P = (x, y) \in E(F)_f$ with $P \in I_f$, then $y = 0$, which means that P is a torsion point. So $P = O$ is the infinite point, and $E(F)_f \oplus I_f = E(F)_f + I_f \subset E(H)_f$. For any $Q \in E(H)_f$, we have $2Q = (Q + Q^\sigma) + (Q - Q^\sigma)$, where $\sigma \in G$. Via the definition of $E(F)_f$ and I_f , we have

$$E(F)_f = \{P | P^\sigma = P, \forall P \in E(H)_f\}, \quad I_f = \{P | P^\sigma = -P, \forall P \in E(H)_f\}.$$

So $Q + Q^\sigma \in E(F)_f$, $Q - Q^\sigma \in I_f$, $2Q \in E(F)_f \oplus I_f$. Thus $2E(H)_f \subset E(F)_f \oplus I_f \subset E(H)_f$. This completes the proof.

As for the index of $E(F)_f \oplus I_f$ in $E(H)_f$, we have the following theorem.

Theorem 1.

$$(E(H)_f : E(F)_f \oplus I_f) = \frac{2^l}{|H^1(G, E(H)_f)|},$$

where $|H^1(G, E(H)_f)|$ is the order of the cohomology group $H^1(G, E(H)_f)$.

Proof. Consider $H^1(G, E(H)_f) = Z^1(G, E(H)_f)/B^1(G, E(H)_f)$. Let $T = \{P - P^\sigma | P \in E(H)_f, \sigma \in G\}$. we will prove that $Z^1(G, E(H)_f) \cong I_f$, $B^1(G, E(H)_f) \cong T$.

For any cocycle $\xi \in Z^1(G, E(H)_f)$, let $\xi \xrightarrow{\phi} \xi_\sigma$. By the definition of cocycle we have that $0 = \xi_e = \xi_{\sigma^2} = (\xi_\sigma)^\sigma + \xi_\sigma$, so $(\xi_\sigma)^\sigma = -\xi_\sigma$, thus $\xi_\sigma \in I_f$, and ϕ is a map of $Z^1(G, E(H)_f)$ to I_f . Via the map ϕ we could see that $Z^1(G, E(H)_f) \cong I_f$, $B^1(G, E(H)_f) \cong T$. Now consider the homomorphism $E(H)_f \xrightarrow{\psi=P-P^\sigma} T$. Obviously $2I_f \subset T$. Since $\psi^{-1}(2I_f) = E(F)_f \oplus I_f$, so

$$(E(H)_f : E(F)_f \oplus I_f) = (T : 2I_f) = (I_f : 2I_f)/(I_f : T),$$

$$= 2^l / |H^1(G, E(H)_f)|.$$

3 Main Results and Their Proofs

We first give a general theorem on torsion-free finitely-generated module over Dedekind domain, which establishes a relationship between the Steinitz class and the index of the module in its corresponding free module. This theorem is the key to our final results about Steinitz class.

Theorem 2. Suppose that L is a free \mathcal{O}_K -module, and $M \subset L$ is a submodule, $(L : M) < +\infty$. Then there is an integral \mathcal{O}_K -ideal \mathcal{A} such that $[\mathcal{A}]$ is the Steinitz class of M , and $N_{\mathbf{Q}}^K(\mathcal{A}) = (L : M)$, where $N_{\mathbf{Q}}^K(\cdot)$ is the norm map of ideals from K to the rationals \mathbf{Q} .

Proof. Let $L = \bigoplus_{i=1}^n \mathcal{O}_K e_i$, so $\{e_1, \dots, e_n\}$ is an \mathcal{O}_K -basis for L . We will inductively prove that there is \mathcal{O}_K -ideals \mathcal{B}_i ($i = 1, \dots, n$) such that $M \cong \bigoplus_{i=1}^n \mathcal{B}_i$, and $(L : M) = \prod_{i=1}^n (\mathcal{O}_K : \mathcal{B}_i)$.

When $n = 1$, every thing is obvious. Assume then the statement is true for $n - 1$ and consider the homomorphism of \mathcal{O}_K -modules: $\rho : L \rightarrow \mathcal{O}_K$, $\rho(\sum_{i=1}^n r_i e_i) = r_n$. Then $\mathcal{B} = \rho(M)$ is an ideal of \mathcal{O}_K , and the sequence

$$0 \rightarrow N \rightarrow M \xrightarrow{\rho} \mathcal{B} \rightarrow 0$$

is exact, where $N = \ker(\rho) \cap M$. Since \mathcal{B} is a projective \mathcal{O}_K -module, there exists \mathcal{O}_K -module $\mathcal{C} \subset M$ such that $\mathcal{C} \cong \mathcal{B}$, $\rho(\mathcal{C}) = \mathcal{B}$, $M = N \oplus \mathcal{C} \cong N \oplus \mathcal{B}$. Thus

$$(L : M) = (L : N \oplus \mathcal{C}) = (L : \bigoplus_{i=1}^{n-1} \mathcal{O}_K + \mathcal{C})(\bigoplus_{i=1}^{n-1} \mathcal{O}_K + \mathcal{C} : N \oplus \mathcal{C})$$

where $(L : \bigoplus_{i=1}^{n-1} \mathcal{O}_K + \mathcal{C}) = (\rho^{-1}(\mathcal{O}_K)) : \rho^{-1}(\mathcal{B}) = (\mathcal{O}_K : \mathcal{B})$.

Consider $\mathcal{C} \cap \bigoplus_{i=1}^{n-1} \mathcal{O}_K = \mathcal{C} \cap \ker(\rho)$. When restricted on \mathcal{C} , the map ρ is injective, so we have

$$\bigoplus_{i=1}^{n-1} \mathcal{O}_K + \mathcal{C} = \bigoplus_{i=1}^{n-1} \mathcal{O}_K \oplus \mathcal{C},$$

$$\begin{aligned}
(\bigoplus_{i=1}^{n-1} \mathcal{O}_K + \mathcal{C} : N \oplus \mathcal{C}) &= (\bigoplus_{i=1}^{n-1} \mathcal{O}_K \oplus \mathcal{C} : N \oplus \mathcal{C}) \\
&= (\bigoplus_{i=1}^{n-1} \mathcal{O}_K : N).
\end{aligned}$$

Note that $N \subset \bigoplus_{i=1}^{n-1} \mathcal{O}_K$. So via the hypothesis of our induction, we know that there are \mathcal{O}_K -ideals \mathcal{B}_i ($i = 1, \dots, n-1$) such that $N \cong \bigoplus_{i=1}^{n-1} \mathcal{B}_i$, and $(\bigoplus_{i=1}^{n-1} \mathcal{O}_K : N) = \prod_{i=1}^{n-1} (\mathcal{O}_K : \mathcal{B}_i)$. Thus we have $M \cong \bigoplus_{i=1}^n \mathcal{B}_i$ and $(L : M) = \prod_{i=1}^n (\mathcal{O}_K : \mathcal{B}_i) = \prod_{i=1}^n N_{\mathbf{Q}}^K(\mathcal{B}_i) = N_{\mathbf{Q}}^K(\prod_{i=1}^n \mathcal{B}_i)$, where $\mathcal{B}_n = \mathcal{B}$. Now the proof is completed by the following lemma.

Lemma 5. Assume \mathcal{A}_1 and \mathcal{A}_2 are two non-zero ideals of Dedekind domain R , then we have isomorphism of R -modules : $\mathcal{A}_1 \oplus \mathcal{A}_2 \cong R \oplus \mathcal{A}_1 \mathcal{A}_2$.

Proof. See Lemma 13 in [7] p.168.

We now intend to prove our main results via our Theorem 2. First we need to find what the L and M of Theorem 2 correspond to in $E(H)$.

Lemma 6. $L = \mathcal{O}_K \cdot E(F)_f$ is a free \mathcal{O}_K -module of rank l .

Proof. Assume P_1, \dots, P_l form a \mathbf{Z} -basis of $E(F)_f$. We will prove

$$L = \mathcal{O}_K \cdot E(F)_f = \bigoplus_{i=1}^l \mathcal{O}_K P_i.$$

Now suppose that $\sum_{i=1}^l [\alpha_i] P_i = 0$ for some $\alpha_i \in \mathcal{O}_K$ ($i = 1, \dots, l$). When $D \equiv 3 \pmod{4}$, we have $\alpha_i = s_i + t_i(1 + \sqrt{-D})/2$ ($s_i, t_i \in \mathbf{Z}$, $i = 1, \dots, l$), then via $\sum_{i=1}^l [\alpha_i] P_i = 0$ we have $\sum_{i=1}^l [2s_i + t_i] P_i = 0$ and $\sum_{i=1}^l [\sqrt{-D} t_i] P_i = 0$. Thus $t_i = 0$, $s_i = 0$, $\alpha_i = 0$ ($i = 1, \dots, l$). This proves the theorem when $D \equiv 3 \pmod{4}$. The case $D \equiv 1 \pmod{4}$ goes in the same way.

The corresponding free modules F for M varies for different D . First consider the case $D \equiv 3 \pmod{4}$.

Theorem 3. For $D \equiv 3 \pmod{4}$, we have $|H^1(G, E(H)_f)| = 1$, and $E(H)_f = \mathcal{O}_K \cdot E(F)_f + I_f$.

Proof. Let P_1, \dots, P_l form a \mathbf{Z} -basis of $E(F)_f$, and Q_1, \dots, Q_l form a \mathbf{Z} -basis of I_f . Put $\alpha = (1 + \sqrt{-D})/2$. We need only to prove that $E(H)_f/(E(F)_f \oplus I_f) = C_1 \oplus \dots \oplus C_l$, where $C_i = (\overline{[\alpha]P_i})$ is subgroup of order 2 generated by $\overline{[\alpha]P_i}$ in the quotient group $E(H)_f/(E(F)_f \oplus I_f)$. Obviously we have $\overline{[\alpha]P_i} \neq \bar{0}$; otherwise there would be $t_j, s_j \in \mathbf{Z}$ ($j = 1, \dots, l$) such that $[\alpha]P_i = \sum_{j=1}^l [t_j]P_j + \sum_{j=1}^l [s_j]Q_j$, then $[1 + \sqrt{-D}]P_i = \sum_{j=1}^l [2t_j]P_j + \sum_{j=1}^l [2s_j]Q_j$, and $P_i = \sum_{j=1}^l [2t_j]P_j$, giving a contradiction.

Furthermore, if $\sum_{i=1}^l [u_i]\overline{[\alpha]P_i} = 0$ for some $u_i \in \mathbf{Z}$ ($i = 1, \dots, l$), then there are $t_i, s_i \in \mathbf{Z}$ ($i = 1, \dots, l$) such that $\sum_{i=1}^l [u_i\alpha]P_i = \sum_{i=1}^l [t_i]P_i + \sum_{i=1}^l [s_i]Q_i$, so

$$\sum_{i=1}^l [u_i]P_i + \sum_{i=1}^l [u_i\sqrt{-D}]P_i = \sum_{i=1}^l [2t_i]P_i + \sum_{i=1}^l [2s_i]Q_i.$$

Thus $\sum_{i=1}^l [u_i]P_i = \sum_{i=1}^l [2t_i]P_i$, which gives $u_i = 2t_i$ ($i = 1, \dots, l$). Hence $[u_i]\overline{[\alpha]P_i} = \overline{[t_i][2\alpha]P_i} = \overline{[t_i(1 + \sqrt{-D})]P_i} = \bar{0}$. This completes the proof.

Now we could prove our main results via Theorem 2.

Theorem 4. Suppose that $D = p \equiv 3 \pmod{4}$ is a prime number, and elliptic curve E has complex multiplication by the full ring \mathcal{O}_K of integers of $K = \mathbf{Q}(\sqrt{-D})$. Then the Steinitz class of E is the principal class, i.e. $St(E) = 1$.

Proof. Let $L = \mathcal{O}_K \cdot E(F)_f$, $M = [\sqrt{-p}]E(H)_f$. Since $M \cong E(H)_f$, so we need only to prove $St(M)$ is the principal class.

By Theorem 3 we have $E(H)_f = \mathcal{O}_K \cdot E(F)_f + I_f$. Thus

$$M = [\sqrt{-p}]E(H)_f = E(F)_f \cdot (\sqrt{-p}\mathcal{O}_K) + [\sqrt{-p}]I_f \subset \mathcal{O}_K \cdot E(F)_f = L;$$

$$\begin{aligned} (L : M) &= (\mathcal{O}_K \cdot E(F)_f : [\sqrt{-p}]E(H)_f) \\ &= \frac{(E(H)_f : [\sqrt{-p}]E(H)_f)}{(E(H)_f : \mathcal{O}_K \cdot E(F)_f)} \\ &= \frac{p^l}{(E(H)_f : \mathcal{O}_K \cdot E(F)_f)}. \end{aligned}$$

Since p is a prime number, so there is t ($0 \leq t \leq l$) such that $(L : M) = p^t$. By Theorem 2, the Steinitz class of M is equal to $[\mathcal{A}]$ for some \mathcal{O}_K -ideal \mathcal{A} , and $p^t = (L : M) =$

$N_{\mathbf{Q}}^K(\mathcal{A})$. Since p is a prime number, $\mathcal{A} = (\sqrt{-p}\mathcal{O}_K)^t$ is principal. Thus $St(E) = St(M)$ is the principal class.

Theorem 5. Suppose that $D = p \equiv 3 \pmod{4}$ is a prime number, and E is an elliptic curve having complex multiplication by the ring \mathcal{O}_K of all integers of $K = \mathbf{Q}(\sqrt{-D})$. Then the Steinitz class of E is $St(E) = [\mathcal{P}^t]$, where $[\mathcal{P}]$ is the ideal class of K represented by \mathcal{P} the prime factor of 2 in \mathcal{O}_K , $2^t = 2^l |H^1(G, E(H)_f)|$. In particular, the parity of t determines $St(E)$ since \mathcal{P} is not principal while $\mathcal{P}^2 = 2\mathcal{O}_K$ is.

Proof. Let $L = \mathcal{O}_K \cdot E(F)_f$, $M = [2\sqrt{-p}]E(H)_f$. Since $M \cong E(H)_f$, so $St(E) = St(M)$. Note that $[2\sqrt{-p}]E(H)_f \subset [\sqrt{-p}](E(F)_f \oplus I_f)$, $[\sqrt{-p}]I_f \subset E(F)_f$. Thus we have $M \subset \mathcal{O}_K \cdot E(F)_f = L$, and

$$\begin{aligned}
(L : M) &= (\mathcal{O}_K \cdot E(F)_f : [2\sqrt{-p}]E(H)_f) \\
&= \frac{(E(H)_f : [2\sqrt{-p}]E(H)_f)}{(E(H)_f : \mathcal{O}_K \cdot E(F)_f)} \\
&= \frac{(4p)^l}{(E(H)_f : E(F)_f \oplus I_f)(E(F)_f \oplus I_f : \mathcal{O}_K \cdot E(F)_f)} \\
&= \frac{(4p)^l}{2^l |H^1(G, E(H)_f)|^{-1} (I_f : [\sqrt{-p}]E(F)_f)} \\
&= 2^l |H^1(G, E(H)_f)| \cdot p^l / (I_f : [\sqrt{-p}]E(F)_f).
\end{aligned}$$

Thus $(L : M) = 2^t p^r$ for some $t, r \geq 0$ since p is a prime number. By Theorem 2 we know that $N_{\mathbf{Q}}^K(\mathcal{A}) = 2^t p^r$ for some \mathcal{O}_K -ideal \mathcal{A} . Therefore $\mathcal{A} = \mathcal{P}^t([\sqrt{-p}]\mathcal{O}_K)^r$, $St(E) = [\mathcal{A}] = [\mathcal{P}^t]$. This proves the theorem.

Corollary 1. Suppose as in Theorem 5. If $l = \text{rank}_{\mathbf{Z}}(E(F)) = 1$ then $H^1(G, E(H)_f)$ determines the Steinitz class of E .

Now we analysis the examples of Dummit and Miller in [5] by utilizing our above method . For these examples, we have $K = \mathbf{Q}(\sqrt{-10})$, $D = 10$, $H = K(\sqrt{5}) = \mathbf{Q}(\sqrt{-10}, \sqrt{5})$. We consider the \mathcal{O}_K -module $L = \mathcal{O}_K \cdot E(F)_f$ and $M = 2[\sqrt{-10}]E(H)_f$. Then

$$(L : M) = \frac{(E(H)_f : 2[\sqrt{-10}]E(H)_f)}{(E(H)_f : \mathcal{O}_K \cdot E(F)_f)}$$

$$\begin{aligned}
&= \frac{(4 \cdot 10)^l}{(E(H)_f : E(F)_f \oplus I_f)(E(F)_f \oplus I_f : \mathcal{O}_K \cdot E(F)_f)} \\
&= \frac{(40)^l}{2^l |H^1(G, E(H)_f)|^{-1} (I_f : [\sqrt{-10}]E(F)_f)} \\
&= 2^l |H^1(G, E(H)_f)| 10^l / (I_f : [\sqrt{-10}]E(F)_f).
\end{aligned}$$

Thus the Steinitz class of E is determined by the 2-exponent of $2^l |H^1(G, E(H)_f)| (I_f : [\sqrt{-10}]E(F)_f)$.

(DM1) Consider the following elliptic curve of Dummit and Miller:

$$E_1 : y^2 = x^3 + (6 + 6\sqrt{5})x^2 + (7 - 3\sqrt{5}).$$

Then $l = 1$, $|H^1(G, E(H)_f)| = 2$, $(I_f : [\sqrt{-10}]E(F)_f) = 1$, so $2^l |H^1(G, E(H)_f)| (I_f : [\sqrt{-10}]E(F)_f) = 4$. Thus the Steinitz class of E_1 is the principal class, i.e., $St(E_1) = 1$.

(DM2) Consider the elliptic curve $E_{1, \text{isog}} : y^2 = x^3 - 912 + 12\sqrt{5}x^2 + (188 + 84\sqrt{5})x$ in [5]. We have $l = 1$, $|H^1(G, E(H)_f)| = 2$, $(I_f : [\sqrt{-10}]E(F)_f) = 2$, $2^l |H^1(G, E(H)_f)| (I_f : [\sqrt{-10}]E(F)_f) = 2^3$. Thus the Steinitz class $St(E_{1, \text{isog}}) = [\mathcal{P}]$, where \mathcal{P} is a prime factor of 2 in \mathcal{O}_K .

(DM3) For $E_3 : y^2 = x^3 + 612x^2 + (46818 - 20808\sqrt{5})x$ in [5], we have $l = 2$, $|H^1(G, E(H)_f)| = 2$, $(I_f : [\sqrt{-10}]E(F)_f) = 1$, $2^l |H^1(G, E(H)_f)| (I_f : [\sqrt{-10}]E(F)_f) = 2^3$. Thus $St(E_3) = [\mathcal{P}]$, \mathcal{P} a prime factor of 2 in \mathcal{O}_K .

There are still many open problems about the Steinitz classes of elliptic curves having complex multiplication. For example, in the case $K = \mathbf{Q}(\sqrt{-D})$ with prime $D \equiv 1 \pmod{4}$, we have the following conjecture.

Conjecture. Both the cases $St(E) = 1$ and $St(E) \neq 1$ exist for some elliptic curves E having complex multiplication by \mathcal{O}_K with $K = \mathbf{Q}(\sqrt{-D})$ mentioned above.

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